MATRICES

Proposition.

Let n and p be two non-zero natural numbers, E and F two finite-dimensional vector spaces over $\mathbb{R}$ such that $\dim(E) = p$ and $\dim(F) = n$.

Let $B = (e_1, \ldots, e_p)$ be a basis of E and $B' = (\varepsilon_1, \ldots, \varepsilon_n)$ a basis of F.

Consider $f \in \mathcal{L}(E, F)$ and $A = Mat_{B,B'}(f) = (a_{i,j})$.

For all $u \in E$, let X denote the column matrix containing the coordinates of u in basis B and Y the coordinates of $f(u)$ in basis B'.

Then,

$$Y = AX$$

Remarks :

1. Note that $Y = AX \iff \mathrm{Mat}_{B'}\big(f(u)\big) = \mathrm{Mat}_{B,B'}(f) \times \mathrm{Mat}_B(u)$
2. $A = \mathrm{Mat}_{B,B'}(f) = (a_{i,j})$. The columns of A contain the coordinates of the vectors $f(e_j)$ in basis B' :

$$\begin{array}{c} \begin{array}{ccc} f(e_1) & & f(e_p) \\ \downarrow & & \downarrow \end{array} \\ \begin{pmatrix} a_{1,1} & \ldots & a_{1,p} \\ \vdots & & \vdots \\ a_{n,1} & \ldots & a_{n,p} \end{pmatrix} \end{array} \begin{array}{l} \\ \leftarrow \text{coefficient of } \varepsilon_1 \\ \\ \leftarrow \text{coefficient of } \varepsilon_n \end{array}$$

   that is, $\forall\, j \in [\![1, p]\!],\ f(e_j) = a_{1,j}\varepsilon_1 + \ldots + a_{n,j}\varepsilon_n$.
3. $A \in \mathcal{M}_{n,p}(\mathbb{R})$, $X \in \mathcal{M}_{p,1}(\mathbb{R})$ and $Y \in \mathcal{M}_{n,1}(\mathbb{R})$.

Proof :

Let $u \in E$.

• Since B is a basis of E :

$$\exists!\ (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p,\ u = \lambda_1 e_1 + \ldots + \lambda_p e_p$$

Then $X = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix}$.

• Using the linearity of f, we get :

$$\begin{aligned} f(u) &= \lambda_1 f(e_1) + \ldots + \lambda_p f(e_p) \\ &= \lambda_1 \left(a_{1,1}\varepsilon_1 + \ldots + a_{n,1}\varepsilon_n\right) + \ldots + \lambda_p \left(a_{1,p}\varepsilon_1 + \ldots + a_{n,p}\varepsilon_n\right) \\ &= \left(\lambda_1 a_{1,1} + \ldots + \lambda_p a_{1,p}\right)\varepsilon_1 + \ldots + \left(\lambda_1 a_{n,1} + \ldots + \lambda_p a_{n,p}\right)\varepsilon_n \end{aligned}$$

It follows that $Y = \begin{pmatrix} \lambda_1 a_{1,1} + \ldots + \lambda_p a_{1,p} \\ \vdots \\ \lambda_1 a_{n,1} + \ldots + \lambda_p a_{n,p} \end{pmatrix} = \begin{pmatrix} a_{1,1} & \ldots & a_{1,p} \\ \vdots & \ldots & \vdots \\ a_{n,1} & \ldots & a_{n,p} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix} = AX$